The Husain-Kuchař Model: Time Variables and Non-degenerate Metrics

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ABSTRACT

We study the Husain-Kuchař model by introducing a new action principle similar to the self-dual action used in the Ashtekar variables approach to Quantum Gravity. This new action has several interesting features; among them, the presence of a scalar time variable that allows the definition of geometric observables without adding new degrees of freedom, the appearance of a natural non-degenerate four-metric and the possibility of coupling ordinary matter.

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I Introduction

In the long quest to understand General Relativity (G.R.) the use of toy models has a long tradition. This is especially true in Quantum Gravity and Quantum Cosmology where they have allowed to obtain some, otherwise very difficult to get, information. However, this does not come without a price because one is usually forced to introduce very strong simplifying assumptions and, quite often, some of the key features of the theory are lost. Though a final judgement on the success of this approach can only be made once a consistent Quantum Gravity theory is found, it is possible, in principle, to get some clues on how well one is doing by considering widely different toy models.

Bianchi models (see, for example, [1]) are obtained by imposing homogeneity conditions on the gravitational variables. Their high symmetry has the consequence of killing most of the degrees of freedom of the full theory leaving only a finite number of them. They have been widely used in Quantum Cosmology mainly because the equations obtained upon quantization are more or less tractable.

There are other (less known) toy models that achieve the goal of simplifying the theory by going in the opposite direction: *adding* degrees of freedom. Chief among them is the Husain-Kuchař model [2] (H-K in the following). This model is quite interesting because it has some of the features that make G.R. so difficult to deal with in the quantum regime, in particular diffeomorphism invariance, but is significantly simpler because it lacks the Hamiltonian constraint (another important source of difficulties in full G.R.). This has the effect of *increasing* the number of degrees of freedom per space point from two to three.

To illustrate with a picture the different and complementary roles played by these two approaches one can make the following analogy: Portray G.R. as a complicated, knotted, two-dimensional-surface Σ embedded in $\mathbb{R}^3$. Working with Bianchi models is something akin to trying to get information about Σ by looking at a finite number of points on it. The H-K model, on the other hand, is like trying to gather information by studying the whole $\mathbb{R}^3$. Clearly some crucial features are lost in both approaches

but, still, they provide useful and complementary views about Σ.[1]

The H-K model, in its usual formulation (see [3] for some alternative descriptions), can be conveniently derived from an action principle very close to the self-dual action [4] from which the Ashtekar approach to classical and quantum G.R. [5] can be found. The phase space of the Hamiltonian description of both theories is the same: it is coordinatized by a $SO(3)$ connection and a densitized (inverse) triad canonically conjugate to it. Their crucial difference is the absence of a Hamiltonian constraint in the H-K model. The usual interpretation of this lack of "dynamics" is the following: By using the frame field in terms of which the H-K action is written[2] one can build a degenerate four-metric g_{ab} and a densitized vector field $\tilde{n}^a$ (that can be de-densitized by means of an auxiliary space-time foliation). The lack of dynamics can be seen as the fact that the Lie derivative of g_{ab} in the direction of n^a is zero.

The four-dimensional metric that we can build from the frame field in the H-K action is degenerate. This can lead to the erroneous conclusion that the model describes *only* degenerate four-metrics; a fact that has induced some authors to claim, for example, that ordinary matter cannot be coupled to the model. We will show that this is not the case in due time but at this point we urge the reader to think about the following paradoxical situation: The fact that the Hamiltonian constraint is missing from the H-K model means that the constraint hypersurface of G.R. in the Ashtekar formulation is contained in the H-K one, hence, every solution to G.R (for example Minkowski space-time) is a solution to the H-K model. How can we then describe these G.R. solutions in terms of the fields present in the H-K action if we only have a 4×3 frame field available?

The solution to this problem that we give in the paper has some unexpected implications that make it quite attractive. On one hand it provides an elegant way to define quantum geometric observables (such as areas and volumes) without having to resort to increasing the number of physical degrees of freedom as in previous approaches [6],[7]. On the other, it allows the introduction of a kind of time variable in

[1] This analogy is, actually, a little bit more than that because the Hamiltonian formulation of G.R. can be understood as the study, in phase space, of the hypersurface defined by the constraints.

[2] Being a 4×3 matrix it is neither a tetrad nor a triad!

the double sense that dynamics can be referred to it and also that the scalar constraint (that we need now in order to get the correct counting of degrees of freedom) is linear in its canonically conjugate momentum (so that, upon quantization it gives a Schrödinger type of equation).

The main result of the paper is that it is possible to obtain the H-K model from an action principle (also related to the self-dual action) that admits an interpretation in terms of non-degenerate four dimensional metrics. This is achieved by introducing a scalar field that can be interpreted, in a sense that will be made more precise later, as the time variable mentioned before. This will not only solve the paradox presented above but also will provide a means to couple ordinary matter thus enhancing the usefulness of H-K as a toy model. We hope that the possible interpretation of this scalar field as time will help to shed some light on the problem of time in full G.R.

The paper is organized as follows. This introduction is followed by section II where the usual formulation of the Husain-Kuchař model is briefly reviewed. The new action principle, that is the object of this paper, is introduced in section III where we derive it from the well known self-dual action for G.R. The details of the Hamiltonian formulation of our model are spelled out in section IV. There we thoroughly study the derivation of the constraints of the theory and discuss their interpretation. In section V we compare the field equations in both the usual and the new formulation for the H-K model in order to show that they are not in contradiction (a non-trivial fact as the number of equations is different in both cases). Section VI gives a different proof of the equivalence of our "non-degenerate" formulation and the usual one at the Lagrangian level. We also show that the addition of a cosmological constant (made possible in our scheme by the availability of a non-degenerate four-metric) does not lead us beyond the H-K model. We end the paper with section VII, where we give our conclusions and general comments, and an appendix that contains some details of the computations needed to disentangle the constraints in our formulation.

II The Husain-Kuchař Model: A Brief Review

We review in this section the H-K model in its usual formulation in order to describe its main features and collect the most important formulas for future reference. We start from the action [2]

$$S = \frac{1}{2}\int_{\mathcal{M}} d^4x\ \tilde{\eta}^{abcd}\epsilon_{ijk}e_a^i e_b^j F_{cd}^k \tag{1}$$

where our notation is the following: $\mathcal{M}$ is a four-dimensional manifold $\mathcal{M} = \mathbb{R}\times\Sigma$ with Σ a three-dimensional manifold (that we take compact and without boundary so that we can freely integrate by parts). Curved space-time indices are represented by lower case Latin letters from the beginning of the alphabet. We will make no distinction between 4-dimensional and 3-dimensional indices. The dimensionality of a certain field will be clear from the context. The three and four dimensional Levi-Civita tensor densities will be denoted as $\tilde{\eta}^{abc}$ and $\tilde{\eta}^{abcd}$ respectively ($\underset{\sim}{\eta}_{abc}$ and $\underset{\sim}{\eta}_{abcd}$ are their inverses). We use the convention of representing the density weights of geometrical objects by using tildes above (positive) and below (negative) the stem letter representing them. Internal $SO(3)$ indices, running from 1 to 3 will be denoted by Latin letters form the middle of the alphabet and the internal Levi-Civita tensor as ϵ^{ijk}. We will also use a $SO(3)$ connection $A_a^i(x)$ that defines a covariant derivative acting on internal indices as $\nabla_a\lambda_i = \partial_a\lambda_i + \epsilon_{ijk}A_a^j\lambda^k$ and can be extended to space-time indices by using any torsion-free space-time connection; none of the results that we present in the paper will depend on the extension chosen. The curvature of $A_a^i(x)$ is defined as $F_{ab}^i = 2\partial_{[a}A_{b]}^i + \epsilon^i{}_{jk}A_a^j A_b^k$. The frame field e_a^i in the previous action is a 4×3 matrix; we will reserve the name triad for its projection on the 3-dimensional slices used in the Hamiltonian formalism.

The field equations derived from (1) are

$$\begin{aligned}&\epsilon_{ijk}e_{[b}^j F_{cd]}^k = 0\\ &\epsilon_{ijk}e_{[b}^j \nabla_c e_{d]}^k = 0\end{aligned} \tag{2}$$

Some interesting features of (2) are summarized in the following formulas

$$\begin{aligned} &\tilde{n}^a F^i_{ab} = 0 \\ &\tilde{n}^a \nabla_{[a} e^i_{b]} = 0 \\ &\mathcal{L}_{n^a}(e^i_a e_{bi}) = 0 \end{aligned} \tag{3}$$

where $\tilde{n}^a = \frac{1}{3!}\tilde{\eta}^{abcd}\epsilon_{ijk}e^i_b e^j_c e^k_d$, $n^a = \tilde{n}^a/\tilde{e}$, and $\tilde{e}$ is defined by means of an auxiliary foliation defined by a scalar function t as $\tilde{e} \equiv \tilde{n}^a \partial_a t$. $\mathcal{L}_{n^a}$ denotes the Lie derivative along the direction defined by n^a. The first two equations in (3) explain why we do not have a dynamics in the model [2] (the projections of the field equations onto the direction normal to the spatial slices are zero) while the last one, which is a consequence of the others, displays this lack of evolution as the fact that the Lie derivative of the degenerate four-metric $e^i_a e_{bi}$ along n^a is zero.

The meaning of this model is best understood in the Hamiltonian framework. In order to define it we introduce a foliation by means of a scalar function t and a congruence of curves (nowhere tangent to the surfaces of the foliation) parametrized by t whose tangent vectors we denote t^a. By doing this we have that $t^a \partial_a t = 1$ and, hence, the time derivatives can be interpreted as the Lie derivatives along the direction defined by t^a. We can write (1) as

$$S = \int dt \int_\Sigma d^3x \, \left\{ \dot{A}^i_a \left[\tilde{\eta}^{abc}\epsilon_{ijk}e^j_b e^k_c\right] + A^i_0 \nabla_a \left[\tilde{\eta}^{abc}\epsilon_{ijk}e^j_b e^k_c\right] + e^i_0 \left[\tilde{\eta}^{abc}\epsilon_{ijk}e^j_a F^k_{bc}\right]\right\}$$

where the dots denote time derivatives of the fields (Lie derivatives along the direction defined by t^a), $A^i_0 \equiv t^a A^i_a$, and $e^i_0 \equiv t^a e^i_a$. After following the usual Dirac procedure [8] one finds out that the phase space of the model is coordinatized by an $SO(3)$ connection A^i_a and a canonically conjugate densitized triad $\tilde{E}^a_i$. The first class constraints are

$$\begin{aligned} &\nabla_a \tilde{E}^a_i = 0 \\ &\tilde{E}^a_i F^i_{ab} = 0 \end{aligned}$$

The first constraint (Gauss law) generates internal $SO(3)$ rotations whereas the sec-

ond (known as vector constraint) generates spatial diffeomorphisms[3]. As we can see there is no scalar constraint so that we have three degrees of freedom per space point.

III From The Self-Dual Action to the Husain-Kuchař Model

In this section we introduce a modified action principle for the Husain-Kuchař model that allows us to use four dimensional, non-degenerate metrics in order to describe it. We take as the starting point the self-dual action[4] of Samuel, Jacobson, and Smolin [4]

$$S = -\frac{1}{2}\int_{\mathcal{M}} d^4x\ \tilde{\eta}^{abcd} e_a^I e_b^J F_{cdIJ}^{-} \tag{4}$$

where now e_a^I is a genuine tetrad field and $I = 0, \ldots, 3$ are $SO(4)$ indices, F_{ab}^{IJ-} is the curvature of an anti-self-dual connection A_a^{-IJ} defined by $F_{ab}^{IJ-} = 2\partial_{[a} A_{b]}^{-IJ} + 2A_{[a}^{-IK} A_{b]K}^{-\ \ J}$. Following [9] we write

$$A_a^{-IJ} \equiv \begin{bmatrix} 0 & A_a^j \\ -A_a^i & -\epsilon^{ijk} A_{ak} \end{bmatrix} \qquad\qquad e_a^I \equiv \begin{bmatrix} -\frac{1}{2} v_a & e_a^i \end{bmatrix}$$

So that (4) becomes

$$S = \frac{1}{2}\int_{\mathcal{M}} d^4x\ \tilde{\eta}^{abcd} \left[v_a e_b^i F_{cdi} + \epsilon^{ijk} e_{ai} e_{bj} F_{cdk} \right] \tag{5}$$

As we can see the (anti)-self-dual action can be obtained by adding a term involving a 1-form field v_a to the usual Husain-Kuchař action (1). A full discussion of (5) can be found in [9].

In the view of the previous formula it is natural to wonder what happens if instead of taking v_a as a general one-form one considers it to be the gradient of a scalar $\nabla_a \phi$. Do we still have G.R. or something else? Let us consider then the following action

$$\hat{S} = \frac{1}{2}\int_{\mathcal{M}} d^4x\ \tilde{\eta}^{abcd} \left[-e_a^i F_{bci} \nabla_d \phi + \epsilon^{ijk} e_{ai} e_{bj} F_{cdk} \right] \tag{6}$$

[3] Diffeomorphisms are actually generated by a linear combination of the Gauss law and the vector constraint.

[4] We actually use anti-self-dual fields for calculational purposes.

Before attempting to unravel its physical meaning, some preliminary remarks are in order. First of all the action is no longer $SO(4)$ invariant[5] although it is obviously $SO(3)$ invariant. Second, we see now that S is linear in the time derivatives of ϕ so we expect to have a scalar constraint linear in its canonically conjugate momentum (that after quantization will lead to a Schrödinger type of equation). It is natural to wonder if (6) could be an action for gravity (with an explicit time variable given by the scalar field ϕ). The answer turns out to be in the negative though, at the end of the day, one discovers that (6) is still interesting in its own right. In order to check whether (6) describes G.R. or not we consider the field equations coming form (4) (remembering that we take now $e_a^0 = \nabla_a\phi$). The field equation obtained by varying with respect to A_{IJ}^- is

$$\left[\nabla_{[a}\left(e_b^I e_{c]}^J\right)\right]^- = 0 \tag{7}$$

From (7) we find out immediately that A_{IJ}^- is equal to the anti-self-dual part of the $SO(4)$ connection Γ_a^{IJ} compatible with e_a^I defined by

$$\mathcal{D}_a e_b^I = \partial_a e_b^I - \Gamma_{ab}^c e_c^I + \Gamma_a^{IK} e_{bK} = 0 \tag{8}$$

where Γ_{ab}^c is the Christoffel symbol of the four-metric $g_{ab} \equiv e_a^I e_{bI}$. Notice that, generically, the determinant of e_a^I

$$\det e_a^I = \frac{1}{3!}\tilde{\eta}^{abcd}(\nabla_a\phi)\epsilon^{ijk}e_{bi}e_{cj}e_{dk}$$

is different from zero so that we can invert (8) to write Γ_a^{IJ} in terms of e_a^I and its derivatives. By substituting $A_a^{-IJ} = \Gamma_a^{-IJ}[e,\phi]$ back in (4) we get

$$S = \int_{\mathcal{M}} d^4y\,\sqrt{g[e,\phi]}R[e,\phi]$$

where R is the scalar curvature of $g_{ab} \equiv e_a^I e_{bI} = \nabla_a\phi\nabla_b\phi + e_a^i e_{bi}$. If, by choosing e_a^i and ϕ we can generate arbitrary and non-correlated $g_{ab}[e,\phi](x)$ and

$$\delta g_{ab}[e,\phi](x) = \int_{\mathcal{M}} d^4y\,\left[\frac{\delta g_{ab}(x)}{\delta e_c^i(y)}\delta e_c^i(y) + \frac{\delta g_{ab}(x)}{\delta\phi(y)}\delta\phi(y)\right]$$

[5] Because the gradient of a scalar function does not transform as the zero component of a $SO(4)$ vector [9].

then S must be an action for full G.R., otherwise, it is something else. At a certain point with coordinates x it is indeed true that both g_{ab} and δg_{ab} can be chosen to be anything we want. However, it is not clear that the same conclusion is true *for all the points* in a neighborhood of x due to the restrictions that we have imposed to the form of some of the components of the tetrads (in fact the main result of the paper shows that g_{ab} and δg_{ab} *are not* completely arbitrary in all the points of Σ).

IV Hamiltonian Formulation for the New Action

By introducing a foliation as in section II we can write

$$\hat{S} = \int dt \int_\Sigma d^3x \left\{ \dot{A}_a^i \tilde{\eta}^{abc} \left[\epsilon_{ijk} e_b^j e_c^k - e_{bi} \nabla_c \phi \right] + A_0^i \nabla_a \left[\tilde{\eta}^{abc} \left(\epsilon_{ijk} e_b^j e_c^k - e_{bi} \nabla_c \phi \right) \right] + \right.$$

$$\left. + \frac{1}{2} \dot{\phi}\, \tilde{\eta}^{abc} e_a^i F_{bci} + e_0^i \tilde{\eta}^{abc} \left[\epsilon_{ijk} e_a^j F_{bc}^k - \frac{1}{2} F_{abi} \nabla_c \phi \right] \right\} \equiv \int dt\ L(t)$$

We denote $\tilde{\pi}_i^a(x)$, $\tilde{\pi}_i(x)$, $\tilde{\sigma}_i^a(x)$, $\tilde{\sigma}_i(x)$, and $\tilde{p}(x)$ the momenta canonically conjugate to $A_a^i(x)$, $A_0^i(x)$, $e_a^i(x)$, $e_0^i(x)$, and $\phi(x)$ (with Poisson brackets given symbolically by $\{q, p\} = 1$). We find the following primary constraints

$$\tilde{\pi}_i^a + \tilde{\eta}^{abc} \left[e_{bi} \nabla_c \phi - \epsilon_{ijk} e_b^j e_c^k \right] = 0 \tag{9}$$

$$\tilde{\pi}_i = 0 \tag{10}$$

$$\tilde{\sigma}_i^a = 0 \tag{11}$$

$$\tilde{\sigma}_i = 0 \tag{12}$$

$$2\tilde{p} - \tilde{\eta}^{abc} e_a^i F_{bci} = 0 \tag{13}$$

The Hamiltonian and the total Hamiltonian are

$$H = \int_\Sigma d^3x \left\{ e_0^i \tilde{\eta}^{abc} \left[\frac{1}{2} F_{abi} \nabla_c \phi - \epsilon_{ijk} e_a^j F_{bc}^k \right] + A_0^i \nabla_a \left[\tilde{\eta}^{abc} \left(e_{bi} \nabla_c \phi - \epsilon_{ijk} e_b^j e_c^k \right) \right] \right\} \tag{14}$$

$$H_T = H + \int_\Sigma d^3x \left\{ \lambda_a^i \left[\tilde{\pi}_i^a + \tilde{\eta}^{abc} \left(e_{bi} \nabla_c \phi - \epsilon_{ijk} e_b^j e_c^k \right) \right] + \right. \tag{15}$$

$$\left. + \lambda^i \tilde{\pi}_i + \mu_a^i \tilde{\sigma}_i^a + \mu^i \tilde{\sigma}_i + \zeta \left[2\tilde{p} - \tilde{\eta}^{abc} e_a^i F_{bci} \right] \right\}$$

where $\lambda_a^i(x)$, $\lambda^i(x)$, $\mu_a^i(x)$, $\mu^i(x)$, and $\zeta(x)$ are arbitrary (at this stage) Lagrange multipliers. The conservation under the evolution defined by H_T of the primary

constraints (9-13) gives the following secondary constraints

$$\nabla_a \left[\tilde{\eta}^{abc} \left(e^i_b \nabla_c \phi - \epsilon^{ijk} e_{bj} e_{ck}\right)\right] = 0 \tag{16}$$

$$\tilde{\eta}^{abc} \left[\epsilon_{ijk} e^j_a F^k_{bc} - \frac{1}{2} F_{abi} \nabla_c \phi\right] = 0 \tag{17}$$

and the following conditions on the Lagrange multipliers

$$\tilde{\eta}^{abc} \left[\left(\frac{1}{2}\delta_{ik} \nabla_b \phi + \epsilon_{ijk} e^j_b\right) \left(\mu^k_c - \nabla_c e^k_0 - \epsilon^{klm} e_{cl} A_{0m}\right) - \zeta \nabla_b e_{ci} - \epsilon_{ijk} e^j_0 \nabla_b e^k_c\right] = 0 \tag{18}$$

$$\tilde{\eta}^{abc} \left[\left(\frac{1}{2}\delta_{ik} \nabla_b \phi - \epsilon_{ijk} e^j_b\right) \left(\lambda^k_c - \nabla_c A^k_0\right) - \frac{1}{2}\zeta F_{bci} + \frac{1}{2}\epsilon_{ijk} e^j_0 F^k_{bc}\right] = 0 \tag{19}$$

$$\tilde{\eta}^{abc} \left[\left(\mu^i_a - \nabla_a e^i_0 - \epsilon^{ijk} e_{aj} A_{0k}\right) F_{bci} + 2\left(\lambda^i_a - \nabla_a A^i_0\right) \nabla_b e_{ci}\right] = 0 \tag{20}$$

The conservation in time of (16) and (17) does not generate new secondary constraints but only the following conditions on the Lagrange multipliers

$$\tilde{\eta}^{abc} \left\{\nabla_a \left[\left(\frac{1}{2}\delta_{ik} \nabla_b \phi + \epsilon_{ijk} e^j_b\right) \mu^k_c\right] - (\nabla_a \zeta)(\nabla_b e_{ci}) - \right. \tag{21}$$

$$\left. - \left(\frac{1}{2}\delta_{ik} \nabla_a \phi + \epsilon_{ijk} e^j_a\right) \epsilon^k{}_{lm} \lambda^l_b e^m_c\right\} = 0$$

$$\tilde{\eta}^{abc} \left\{\left(\frac{1}{2}\delta_{ik} \nabla_a \phi - \epsilon_{ijk} e^j_a\right) \nabla_b \lambda^k_c - \frac{1}{2}\epsilon_{ijk} \mu^j_a F^k_{bc} + \frac{1}{2} F^i_{ab} \nabla_c \zeta\right\} = 0 \tag{22}$$

In principle, one expects that some combination of the second class constraints will be first class. The way to find out if this is the case is to solve the equations for the Lagrange multipliers. As we show in the appendix it is possible to find μ^i_a from (18) and λ^i_a from (19) and write them in terms of ζ, e^i_a, e^i_0, A^i_a, and A^i_0

$$\mu^i_a = \nabla_a e^i_0 + \epsilon^{ijk} e_{aj} A_{0k} + \underset{\sim}{P}{}_a{}^i{}_b{}^j \tilde{\eta}^{bcd} \left(\zeta \nabla_c e^j_d + \epsilon_{jkl} e^k_0 \nabla_c e^l_d\right) \tag{23}$$

$$\lambda^i_a = \nabla_a A^i_0 - \frac{1}{2} \underset{\sim}{P}{}_b{}^j{}_a{}^i \tilde{\eta}^{bcd} \left(\zeta F_{cdj} - \epsilon_{jkl} e^k_0 F^l_{cd}\right) \tag{24}$$

where $\underset{\sim}{P}{}_a{}^i{}_b{}^j$ (which is calculated in the appendix) satisfies $\tilde{P}^a{}_i{}^b{}_j \underset{\sim}{P}{}_b{}^j{}_c{}^k = \delta^a_c \delta^i_k$. We have made the ansatz that the triad is non-degenerate (and we will continue to do so

throughout the paper). After some tedious algebra it is possible to verify that (20-22) are identically satisfied by the previous μ_a^i and λ_a^i. We leave (some) of the details for the appendix.

We want to stress here the importance of paying attention to the conditions on the Lagrange multipliers that appear in the Hamiltonian analysis. If one knows beforehand what a theory means, one can usually skip the arduous solution of the consistency equations as one does not need to know the explicit form of the Lagrange multipliers once all the first class constraints have been identified. However, it is true, in general, that the Lagrange multiplier equations themselves may imply additional constraints (they are non-homogeneous linear equations) so, if one does not know the meaning of the theory one is dealing with, great attention must be paid to these equations in order to avoid missing some of the constraints and completely fail in the interpretation of the theory.

Substituting (23-24) in H_T we get

$$\begin{aligned} H_T = \int_{\mathcal{M}} d^3x \; \Big\{ & e_0^i \Big[\tilde{\eta}^{abc} \left(\frac{1}{2} F_{abi} \nabla_c \phi - \epsilon_{ijk} e_a^j F_{bc}^k \right) - \nabla_a \tilde{\sigma}_i^a - \epsilon_{ijk} \tilde{\sigma}_l^a \underset{\sim}{P}{}_a{}^l{}_b{}^j \tilde{\eta}^{bcd} \nabla_c e_d^k - \\ & - \frac{1}{2} \left(\tilde{\pi}_j^a + \tilde{\eta}^{abc} (e_{bj} \nabla_c \phi - \epsilon_{jkl} e_b^k e_c^l) \right) \underset{\sim}{P}{}_d{}^m{}_a{}^j \tilde{\eta}^{def} \epsilon_{imn} F_{ef}^n \Big] - \\ & - A_0^i \left(\nabla_a \tilde{\pi}_i^a + \epsilon_{ijk} e_a^j \tilde{\sigma}^{ak} \right) + \lambda_i \tilde{\pi}^i + \mu_i \tilde{\sigma}^i + \\ & + \zeta \Big[2\tilde{p} - \tilde{\eta}^{abc} e_a^i F_{bci} + \tilde{\sigma}_i^a \underset{\sim}{P}{}_a{}^i{}_b{}^j \tilde{\eta}^{bcd} \nabla_c e_{dj} - \\ & - \frac{1}{2} \underset{\sim}{P}{}_b{}^j{}_a{}^i \tilde{\eta}^{bcd} F_{cdj} \left(\tilde{\pi}_i^a + \tilde{\eta}^{aef} (e_{ei} \nabla_f \phi - \epsilon_{ikl} e_e^k e_f^l) \right) \Big] \Big\} \end{aligned} \tag{25}$$

The terms proportional to e_0^i and A_0^i together give a first class Hamiltonian and the terms proportional to ζ, λ_i, and μ_i are first class constraints (each of them). Of course, we have also all the remaining constraints provided by (9-10, 16, 17). The first class constraints $\tilde{\pi}^i = 0$ and $\tilde{\sigma}^i = 0$ imply that A_0^i and e_0^i are arbitrary functions so we can just remove $\tilde{\pi}^i = 0$ and $\tilde{\sigma}^i = 0$ from (25). Furthermore, as now A_0^i, e_0^i, and ζ are arbitrary and H_T is first class, the expressions that they multiply (linear combinations of first and second class constraints) must be first class constraints. In this way we get three sets of first class constraints plus the following independent second class constraints

$$\tilde{\pi}_i^a + \tilde{\eta}^{abc}\left[e_{bi}\nabla_c\phi - \epsilon_{ijk}e_b^j e_c^k\right] = 0 \tag{26}$$

$$\tilde{\sigma}_i^a = 0 \tag{27}$$

These are very easy to deal with. In practice it is enough to remove $\tilde{\sigma}_i^a$ from the first class constraints and write e_a^i in terms of ϕ and $\tilde{\pi}_i^a$ by solving (26)

$$e_a^i = \frac{1}{4\tilde{\tilde{\pi}}}\underset{\sim}{\eta}_{abc}\left\{\pm\left[2\tilde{\tilde{\pi}} - (\tilde{\pi}_l^d\nabla_d\phi)^2\right]^{1/2}\epsilon^{ijk}\tilde{\pi}_j^b\tilde{\pi}_k^c - 2(\tilde{\pi}_k^d\nabla_d\phi)\tilde{\pi}^{bk}\tilde{\pi}^{ci}\right\}$$

where $\tilde{\tilde{\pi}} \equiv \det\tilde{\pi}_i^a$. The final Hamiltonian description is very simple. The phase space is coordinatized by the canonically conjugate pairs $(A_a^i,\, \tilde{\pi}_i^a)$ and $(\phi,\, \tilde{p})$[6] and the first class constraints are

$$\begin{aligned}
&\nabla_a\tilde{\pi}_i^a = 0\\
&\tilde{\pi}_i^b F_{ab}^i + \tilde{p}\nabla_a\phi = 0\\
&\tilde{p} \mp \frac{1}{2}\left[2\tilde{\tilde{\pi}} - (\tilde{\pi}_l^d\nabla_d\phi)^2\right]^{-1/2}\epsilon^{ijk}\tilde{\pi}_i^a\tilde{\pi}_j^b F_{abk} = 0
\end{aligned} \tag{28}$$

They are the Gauss law, that generates $SO(3)$ gauge transformations, the vector constraint that (essentially) generates diffeomorphisms, and a scalar constraint linear in $\tilde{p}$. They are first class constraints. It is convenient to write them in “weighted” form

$$\begin{aligned}
G(N^i) &= \int_\Sigma d^3x\; N^i\nabla_a\tilde{\pi}_i^a\\
V(N^a) &= \int_\Sigma d^3x\; N^a\left(\tilde{\pi}_i^b F_{ab}^i + \tilde{p}\nabla_a\phi\right)\\
S(N) &= \int_\Sigma d^3x\; N\left\{\tilde{p} \mp \frac{1}{2}\left[2\tilde{\tilde{\pi}} - (\tilde{\pi}_l^d\nabla_d\phi)^2\right]^{-1/2}\epsilon^{ijk}\tilde{\pi}_i^a\tilde{\pi}_j^b F_{abk}\right\}
\end{aligned} \tag{29}$$

The three-dimensional diffeomorphisms are generated by the combination of the Gauss law and the vector constraint $D(N^a) \equiv G(N^a A_a^i) - V(N^a)$. We can write

[6] This is the symplectic structure given by the Dirac brackets.

now the constraint algebra

$$
\begin{aligned}
&\Big\{G(N^i), G(M^i)\Big\} = G([N,M]^i) \qquad\quad \text{with} \qquad\quad [N,M]^i \equiv \epsilon^{ijk} N_j M_k \\
&\Big\{G(N^i), V(M^a)\Big\} = 0 \\
&\Big\{G(N^i), S(M)\Big\} = 0 \\
&\Big\{D(N^a), D(M^b)\Big\} = D(-[N,M]^a) \quad \text{with} \quad [N,M]^a \equiv N^b\partial_b M^a - M^b \partial_b N^a \\
&\{D(N^a), S(M)\} = S(-N^a \nabla_a M) \\
&\{S(N), S(M)\} = V\left[(N\partial_a M - M\partial_a N)\frac{4\tilde{\pi}^a_i \tilde{\pi}^{bi}}{2\tilde{\tilde{\pi}} - (\tilde{\pi}^d_l \nabla_d \phi)^2}\right]
\end{aligned}
\tag{30}
$$

Several remarks are now in order. First, we see that the constraints are first class. As we have 20 canonical variables per space point in Σ and seven first class constraints we have three degrees of freedom per space point –one more that in G.R.–. Second, the Poisson bracket of the scalar constraint with itself closes and gives the vector constraint. This is in agreement with what one would expect from the arguments given by Hojman, Kuchař and Teitelboim in [10] where they showed that the algebra of space-time deformations implied a constraint algebra of the type given by (30). Third, the structure of the scalar constraint is quite suggestive; it has two terms, one linear in $\tilde{p}$ and another proportional to the scalar constraint in the Euclidean Ashtekar formulation for G.R. This may signal a previously unnoticed relation between the Husain-Kuchař model and G.R.

From (28-29) we can interpret the model very easily. It is enough to impose the gauge fixing condition $\phi = 0$ (admissible because $\{\phi(x), \tilde{p}(y)\} = \delta^3(x,y)$) to get rid of the scalar constraint and the ϕ dependent part of the vector constraint to recover the constraints of the usual H-K model, namely

$$
\begin{aligned}
&\nabla_a \tilde{\pi}^a_i = 0 \\
&\tilde{\pi}^a_i F^i_{ab} = 0
\end{aligned}
\tag{31}
$$

This means that in our formulation of the model the gauge orbits have one extra dimension so, in rigor, the models are equivalent only modulo gauge transformations.

At this point the reader may have the temptation to think that, after all, it is trivial to add a scalar constraint to (31) in order to have a time variable (just take $\tilde{p} = 0$ and add the term necessary to generate diffeomorphisms on ϕ and $\tilde{p}$ to the vector constraint). The formulation thus obtained is, obviously, equivalent to ours (and can be derived from the action (6) by removing the derivatives of ϕ with an integration by parts). However, it is much less obvious (and less trivial) the fact that with a suitable choice of a scalar constraint one gets, not only a time variable, but also a way to interpret the H.K model as a theory for non-degenerate four-metrics at the Lagrangian level.

V The four-dimensional Picture: Non-degenerate 4-metrics

The four dimensional field equations coming from the action (6) are

$$\tilde{\eta}^{abcd}\left\{F_{abi}\nabla_c\phi - 2\epsilon_{ijk}e_a^j F_{bc}^k\right\} = 0 \tag{32}$$

$$\tilde{\eta}^{abcd}\left\{\nabla_a e_{bi}\nabla_c\phi + 2\epsilon_{ijk}e_a^j\nabla_b e_c^k\right\} = 0 \tag{33}$$

$$\tilde{\eta}^{abcd}(\nabla_a e_b^i)F_{cdi} = 0 \tag{34}$$

If we have a solution to these equations we can build a four-metric from the tetrad given by $(\nabla_a\phi, e_a^i)$ as $g_{ab} = \pm\nabla_a\phi\nabla_b\phi + e_a^i e_{bi}$. Notice that it is possible to write both Euclidean and Lorentzian 4-metrics by choosing the sign in front of the $\nabla_a\phi\nabla_b\phi$ term. In general one expects that g_{ab} is non-degenerate as can be checked by simply showing some solutions to (32-34) such as

$$A_a^i = 0, \qquad \phi = x^0 \qquad e_a^i = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

As it can be seen (35) provides both the Euclidean and the Minkowski metric in $\mathbb{R}^4$. We see that we can solve the (apparent) paradox presented in the introduction by

using the scalar field that is present now in the field equations to build non-degenerate four-metrics.

It is interesting at this point to compare the new equations (32-34) with the old ones (2). For starters we seem to have one more equation now that we had before; however, as we show below, this equation is not independent of the others and, also, any solution to (2) is a solution to it. In the following we use a procedure similar to the one that appears in section III of [2]. Let us write

$$E^i_{ab} \equiv \nabla_{[a} e^i_{b]}$$

$$\tilde{n}^a \equiv \tfrac{1}{3!}\tilde{\eta}^{abcd}\epsilon_{ijk}e^i_b e^j_c e^k_d$$

$$\tilde{\eta}^a_i \equiv -\tfrac{1}{2}\tilde{\eta}^{abcd}\epsilon_{ijk}e^j_b e^k_c \nabla_d \phi$$

Now $\tilde{n}^a\tilde{n}^b E^i_{ab} = 0$ implies that there must exist $\tilde{E}_j{}^i$ such that

$$\tilde{n}^a E^i_{ab} = e^j_b \tilde{E}_j{}^i$$

Notice that e^i_a satisfy $\tilde{n}^a \tilde{E}^i_a = 0$ so that any linear combination of the e^i_a such as $e^j_a \tilde{E}_j{}^i$ will also satisfy $\tilde{n}^a \tilde{E}_j{}^i e^j_a = 0$. By the same reasoning there must exist $\tilde{F}_j{}^i$ such that

$$\tilde{n}^a F^i_{ab} = e^j_b \tilde{F}_j{}^i$$

We define also (we suppose $\tilde{n}^d \nabla_d \phi \neq 0$)

$$e^{kl} \equiv \frac{1}{(\tilde{n}^d \nabla_d \phi)^2}\epsilon^{ijk}\tilde{\eta}^a_i \tilde{\eta}^b_j E^l_{ab}, \qquad f^{kl} \equiv \frac{1}{(\tilde{n}^d \nabla_d \phi)^2}\epsilon^{ijk}\tilde{\eta}^a_i \tilde{\eta}^b_j F^l_{ab}$$

We can extract all the content from the equations (32-34) by multiplying the first two by

$$\epsilon^{ijk}\tilde{\eta}^a_i \tilde{\eta}^b_j \tilde{\eta}^c_k, \qquad \epsilon^{ijk}\tilde{n}^{[a}\tilde{\eta}^b_j \tilde{\eta}^{c]}_k$$

and the scalar equation (34) by

$$\epsilon^{ijk}\tilde{n}^{[a}\tilde{\eta}^b_i \tilde{\eta}^c_j \tilde{\eta}^{d]}_k$$

(which is proportional to $\tilde{\eta}^{abcd}$).

The result that we obtain from (32) is that $F^{ij} = -\frac{1}{2}\left(f^{ij} - \frac{1}{2}\delta^{ij}f\right)$ and f^{ij} is symmetric and from (33) that $E^{ij} = \frac{1}{2}\left(e^{ij} - \frac{1}{2}\delta^{ij}e\right)$ and e^{ij} is symmetric, where e and f are the traces of e_{ij} and f_{ij} respectively. In terms of e^{ij}, f^{ij}, E^{ij}, and F^{ij} the scalar equation (34) gives $e^{ij}F_{ij} + f^{ij}E_{ij} = 0$; we see now that all the solutions to (32) and (33) are solutions to the scalar equation and, hence, it is redundant.

If we consider now the standard H-K equations we see that there is no scalar equation there. It is possible to extract the content of (2) by using the procedure introduced above. The only difference now is that we need an auxiliary scalar function (for example the one that gives the foliation used in the passage to the Hamiltonian formulation) to define $\tilde{\eta}^a_i$. We immediately find the result that appears in [2] $e^{[ij]} = 0$, $f^{[ij]} = 0$, $E^{ij} = 0$, and $F^{ij} = 0$, so that now it is also true that the scalar equation (34) is satisfied. In order to compare the solutions to (2) and (32-34) one must take into account the new symmety present in the model due to the introduction of ϕ.

VI From the Old to the New Husain-Kuchař Model: Equivalence at the Lagrangian Level.

Although we have seen from the Hamiltonian analysis that the new and all formulations of the H-K model are strictly equivalent it is instructive to understand this from an independent point of view because the actions (1) and (6) look quite different (in fact one could claim that (6) is really "closer" to the self-dual action for G.R. than to the H-K action).

The key idea to show this equivalence is the last result of the previous section, i.e. the fact that every solution to the ordinary H-K equations (2) also satisfies (34). This means that nothing changes if we add this condition to the action (1) with a scalar Lagrange multiplier ϕ. In this way we get

$$\hat{S}_1 = \int_{\mathcal{M}} d^4x\, \tilde{\eta}^{abcd}\left[-\phi F^i_{ab}\nabla_c e_{di} + \epsilon_{ijk}e^i_a e^j_b F^k_{cd}\right]$$

which is obviously equivalent to (6). Actually we can go even further. From the H-K equations it is straightforward to show that (2) implies

$$\tilde{\eta}^{abcd}\nabla_a(\epsilon_{ijk}e^i_b e^j_c e^k_d) = 0$$

so that even the action[7]

$$\hat{S}_2 = \int_{\mathcal{M}} d^4x \, \tilde{\eta}^{abcd} \left[\phi F_{ab} \nabla_c e_{di} + \epsilon_{ijk} e^i_a e^j_b F^k_{cd} + \psi \epsilon_{ijk} \nabla_a (e^i_b e^j_c e^k_d) \right]$$

describes the H-K model. This last action admits an interesting interpretation. If we choose $\psi(x) = -\frac{\Lambda}{3!}\phi(x)$ with Λ a real constant and consider the tetrad $e^I_a \equiv (\nabla_a \phi, e^i_a)$ whose inverse is given by

$$e^a_I \equiv \frac{1}{\det e^I_a} \left[\begin{array}{c} \tilde{n}^a \\ \tilde{\eta}^a_i \end{array} \right]$$

with $\tilde{n}^a$ and $\tilde{\eta}^a$ as defined in the previous section we see that the added term is, in fact

$$\int_{\mathcal{M}} d^4x \, \Lambda (\det e^I_a)$$

that is, a cosmological constant term. This is the simplest (and trivial) instance of a matter coupling to the H-K model using the, now available, non-degenerate four metric.

An additional curious fact is that the previous term is equivalent to

$$\int_{\mathcal{M}} d^4x \, \Lambda (\det e^I_a) e^a_J e^{aJ} \nabla_a \phi \nabla_b \phi$$

i.e. the coupling of the ϕ field to the “non-degenerate H-K model” as a free scalar. A Hamiltonian analysis of these last actions with a “cosmological constant” following the lines of section IV shows their equivalence with the usual H-K model.

VII Conclusions and Perspectives

As we have shown in the paper it is possible to describe the Husain-Kuchař model with an action principle for non-degenerate metrics. We have accomplished this by introducing a scalar field in such a way that adds no new degrees of freedom. This scalar plays, in a sense, the role of a time variable not only because we have now a Hamiltonian constraint that is linear in its canonical momentum but also because it allows dynamics to be referred to it. Our proposal should be compared to those of other authors (especially [6] and [7]). In these papers a scalar field is included

[7] There are even more possibilities that we do not discuss here.

as a means to define quantum gauge invariant observables, quoting Rovelli "matter observables which can be used to dynamically determine surfaces, the areas of which, we can measure". Our contribution in this respect is that we have managed to achieve this goal without introducing new degrees of freedom in the model. We find it quite appealing that in this process we get a nice interpretation of the scalar ϕ as time. Not only can we do this but also, as a side result, we have now the possibility of coupling ordinary matter to the model. This provides a type of theories that lie in between those that have a matter evolving in a non-dynamical background and full G.R. We think that a lot can be learned from looking at these theories; we plan to study them in the future. Notice, by the way, that we have the choice of coupling the matter fields to Euclidean or Lorentzian metrics, depending on the choice of the sign in the first term of the four-metric $g_{ab} = \pm\nabla_a\phi\nabla_b\phi + e_a^i e_{bi}$.

We want to remark at this point that not knowing beforehand what the meaning of the action (6) is, one should be very careful in order to avoid missing constraints crucial for the interpretation of the theory. That is why we have paid so much attention to the solution of the equations for the Lagrange multipliers. Also, we emphasize again the contradiction in claiming that the Husain-Kuchař model only allows for the existence of degenerate four-metrics whereas it is obviously an extension of both Euclidean and Lorentzian G.R. We believe that we have clearly solved this seemingly paradoxical fact in the paper.

VIII Appendix

As we have said in the main text of the paper we have paid special attention to the solution of the Lagrange multiplier equations (18-22). The strategy that we have followed is simple. First solve (18) for μ_a^i and (19) for λ_a^i and plug the result in the remaining ones. The result that we have obtained (for non-degenerate triads) shows that once we write these Lagrange multipliers in terms of ζ, A_{0i}, e_{0i}, A_{ai}, e_{ai}, and ϕ the remaining equations are identically satisfied.

In order to solve (18) and (19) we need to compute the inverses (that we denote

$\underset{\sim}{P}{}_a{}^i{}_b{}^j$) of the 9×9 matrices

$$\tilde{P}{}^a_i{}^c_k(x) \equiv \tilde{\eta}^{abc} \left(\frac{1}{2} \delta_{ik} \nabla_b \phi + \epsilon_{ijk} e^j_b \right)$$

and its transpose

$$\tilde{P}{}^c_k{}^a_i(x) \equiv \tilde{\eta}^{abc} \left(-\frac{1}{2} \delta_{ik} \nabla_b \phi + \epsilon_{ijk} e^j_b \right)$$

where ${}^a_i$ are "double indices" that take the nine different values that make these matrices 9×9. The best way to build their inverses is to explicitly solve the equation

$$\tilde{M}{}^a_i{}^b_j X^j_b \equiv \tilde{\eta}^{abc} \left(\delta_{ij} v_c + \epsilon_{ijk} e^k_c \right) X^j_b = \tilde{J}^a_i \tag{36}$$

First we introduce the inverse triad e^a_i such that $e^a_i e^j_a = \delta^j_i$ and write $\tilde{\eta}^{abc} = \tilde{e} \epsilon^{ijk} e^a_i e^b_j e^c_k$ (here $\tilde{e}$ is the non-zero determinant of the triad). Introducing this in (36), expanding, and using the notation

$$X_{ij} \equiv e^a_i X_{aj}, \quad X \equiv e^a_i X^i_a \quad J^a_i \equiv \tilde{J}^a_i / \tilde{e}, \quad J_{ij} \equiv e_{ai} J^a_j, \quad , J \equiv e_{ai} J^{ai}$$

we get

$$\epsilon^{lmn} e^a_l X_{mi} v_n + \epsilon^{lmk} \epsilon_{ijk} e^a_l X_m{}^j = J^a_i$$

which, after multiplying by e_{al} transforms into an equation that only involves objects with internal indices.

$$\epsilon_l{}^{mn} X_{mi} v_n + X \delta_{il} - X_{il} = j_{li} \tag{37}$$

Let us now take the trace of (37), multiply it by $\epsilon_{ilp} v^p$ and by $v_i v_l$. We find the following three equations

$$-\epsilon^{ijk} X_{ij} v_k + 2X = J \tag{38}$$

$$X_{ij} v^i v^j - v^2 X - \epsilon^{ijk} X_{ij} v_k = -\epsilon^{ijk} J_{ij} v_k \tag{39}$$

$$X v^2 - X_{ij} v^i v^j = J_{ij} v^i v^j \tag{40}$$

where $v^2 \equiv v_i v^i$. Adding (39) and (40) and using (38) gives

$$X = \frac{1}{2} \left[J + \epsilon^{ijk} J_{ij} v_k - v^i v^j J_{ij} \right]$$

This means that we know how to express X in (37) in terms of J_{ij} and v_i. If we look now at how the indices in the remaining X_{ij} appear we see that the i index is at both the second and the first place. If we could find the way to have both i indices at the second place the remaining equation would be very easy to solve by inverting a simple 3×3 matrix. To this end we need to know the expression for $X_{[ij]}$ in terms of J_{ij}. This can be computed by multiplying (36) both by v_a and $\epsilon_{ilm} e^m_a$ and eliminating tangent space indices as before. One gets

$$X_{[ij]} = J_{[ij]} - \frac{1}{2}\epsilon_{ijk} J^{lk} v_l$$

Using this result back in (37) we have

$$X^k{}_i(\delta_{jk} - \epsilon_{jkl} v^l) = \frac{1}{2}\delta_{ij}(J + \epsilon^{pqr} J_{pq} v_r - v^p v^q J_{pq}) + \epsilon_{ijk} J^{lk} v_l - J_{ij} \qquad (41)$$

Multiplying (41) by

$$\frac{1}{1+v^2}\left[\delta_{nj} + v_n v_j + \epsilon_{njs} v^s\right]$$

and reintroducing the triads we finally get

$$\underset{\sim}{M}_a{}^i{}_b{}^j = \frac{1}{2\tilde{e}(1+v^2)}\left[\delta_{nl} + v_n v_l + \epsilon_{nlr} v^r\right] \times$$

$$\times \left[\delta^{il}\delta^j_k - 2\delta^i_k \delta^{jl} + \delta^{il}\epsilon^j{}_{mk} v^m + 2\epsilon^{ilj} v_k - v_k v^j \delta^{il}\right] e^n_a e^k_b \qquad (42)$$

The inverses of $\tilde{P}^{\,a}_{\,i}{}^{\,b}_{\,j}$ and its transponse are immediately obtained from the (42). With them it is possible to check by direct substitution that the consistency equations (20-22) are identically satisfied. In practice the best strategy to do this is the following. First, eliminate the tangent space indices by multiplying by suitable combinations of inverse triads, then use the constraints (16) and (17) in the form

$$\left[-\frac{1}{2}\delta_{ik}\nabla_a\phi + \epsilon_{ijk} e^j_a\right]\tilde{\eta}^{abc}\nabla_b e_{ci} = 0$$

$$\left[+\frac{1}{2}\delta_{ik}\nabla_a\phi + \epsilon_{ijk} e^j_a\right]\tilde{\eta}^{abc} F_{bci} = 0$$

In order to check (21) it is very useful to use the following identity

$$\tilde{\eta}^{abc}\left(P^{k}_{ai}\epsilon_{klm}-\hat{P}_{akl}\epsilon^{k}{}_{mi}\right)e^{m}_{c}\Lambda^{l}_{b}=0$$

where

$$P_{aik}\equiv v_a\delta_{ik}-\epsilon_{ijk}e^{j}_{a}$$

$$\hat{P}_{aik}\equiv v_a\delta_{ik}+\epsilon_{ijk}e^{j}_{a}$$

and Λ^{l}_{b} is arbitrary.

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